Module 4: z transform

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- Definition
- Zeros and Poles
- Region of Convergence

The z-Transform

Introduction

Why z-Transform?

- A generalization of Fourier transform
- Why generalize it?
  - FT does not converge on all sequence
  - Notation good for analysis
  - Bring the power of complex variable theory deal with the discrete-time signals and systems

Definition

- The z-transform of sequence $x(n)$ is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

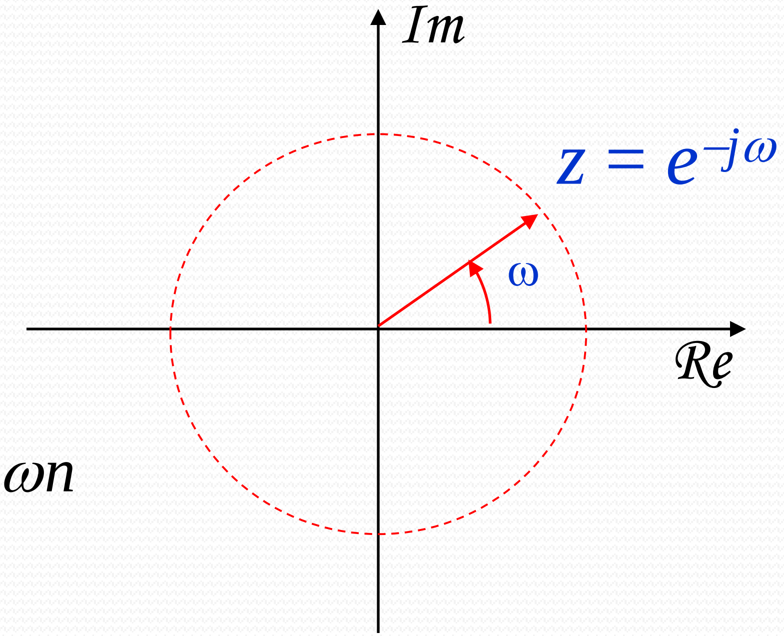
- Let $z = e^{-j\omega}$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

z-Plane

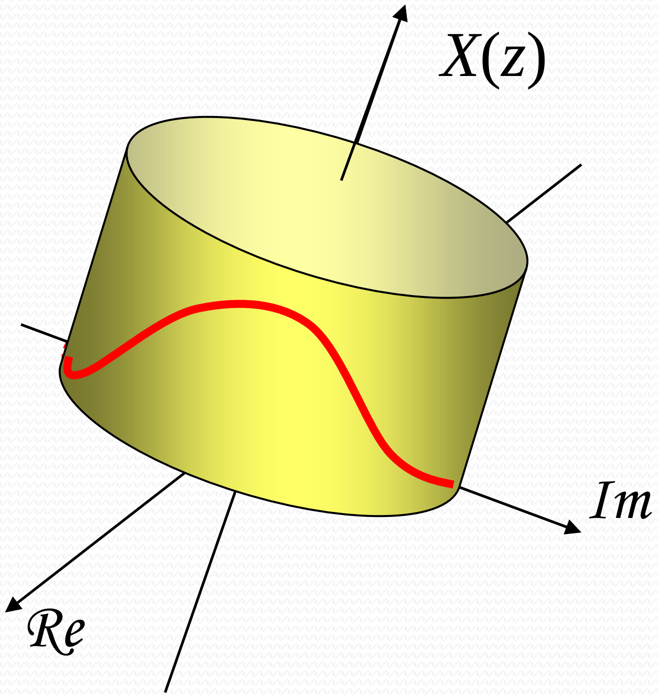
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

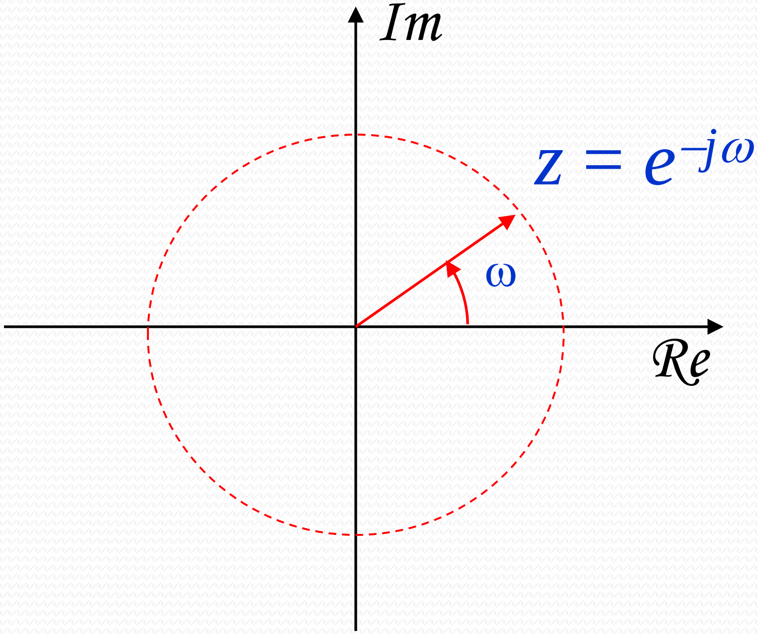
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$



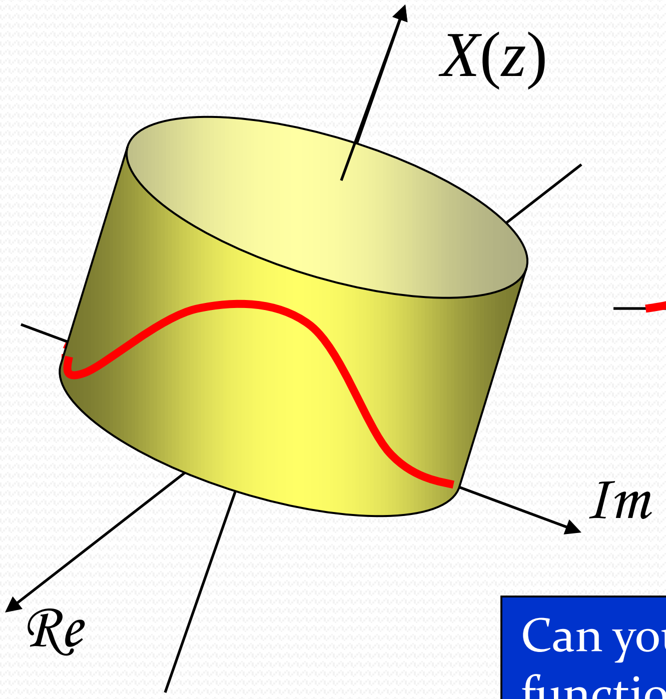
Fourier Transform is to *evaluate z-transform on a unit circle.*

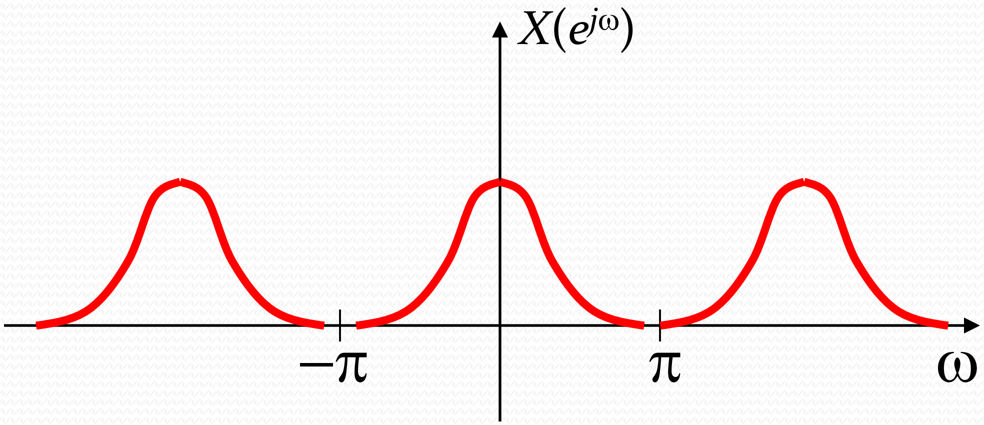
z-Plane





Periodic Property of FT





Can you say why Fourier Transform is a periodic function with period 2π?

The z-Transform

Zeros and Poles

Definition

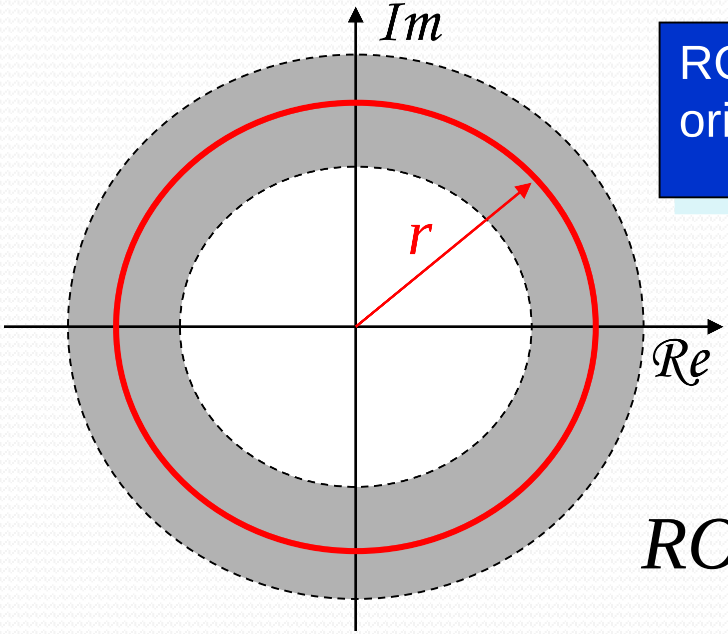
- Give a sequence, the set of values of *z* for which the *z*-transform converges, i.e., $|X(z)|<\infty$, is called the region of convergence.

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n) z^{-n} \right| = \sum_{n=-\infty}^{\infty} |x(n)| |z|^{-n} < \infty$$

ROC is centered on origin and consists of a set of rings.

Example: Region of Convergence

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n) z^{-n} \right| = \sum_{n=-\infty}^{\infty} |x(n)||z|^{-n} < \infty$$



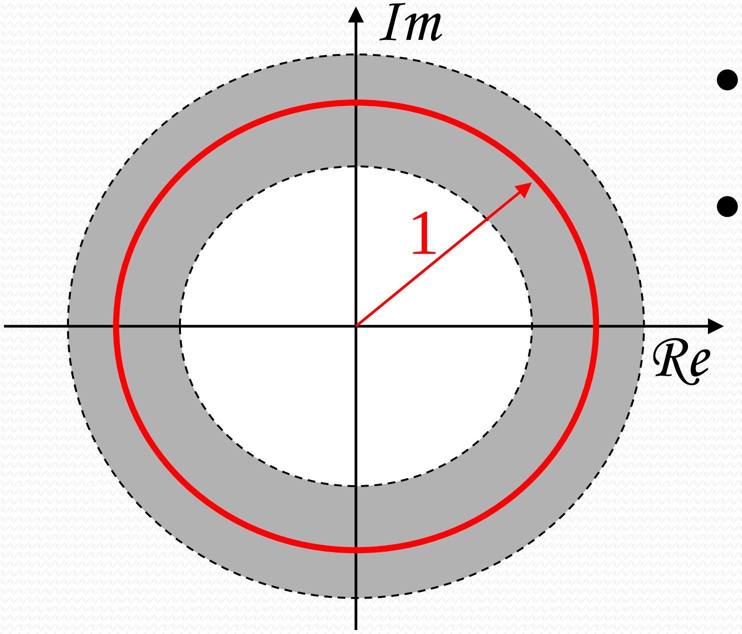
ROC is an annual ring centered on the origin.

$$R_{x-} < |z| < R_{x+}$$

$$ROC = \{ z = re^{j\omega} \mid R_{x-} < r < R_{x+} \}$$

Stable Systems

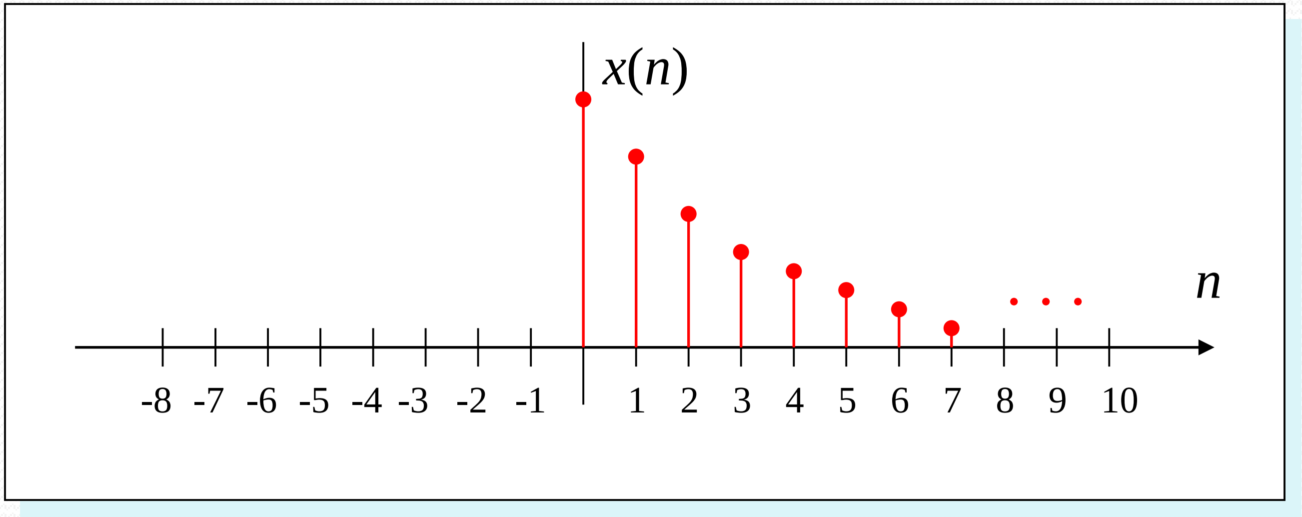
- A stable system requires that its Fourier transform is uniformly convergent.



- Fact: Fourier transform is to evaluate *z*-transform on a unit circle.
- A stable system requires the ROC of *z*-transform to include the unit circle.

Example: A right sided Sequence

$$x(n) = a^n u(n)$$



Example: A right sided Sequence

$$x(n) = a^n u(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} |az^{-1}| < \infty \Rightarrow |az^{-1}| < 1$$

$$\Rightarrow |z| > |a|$$

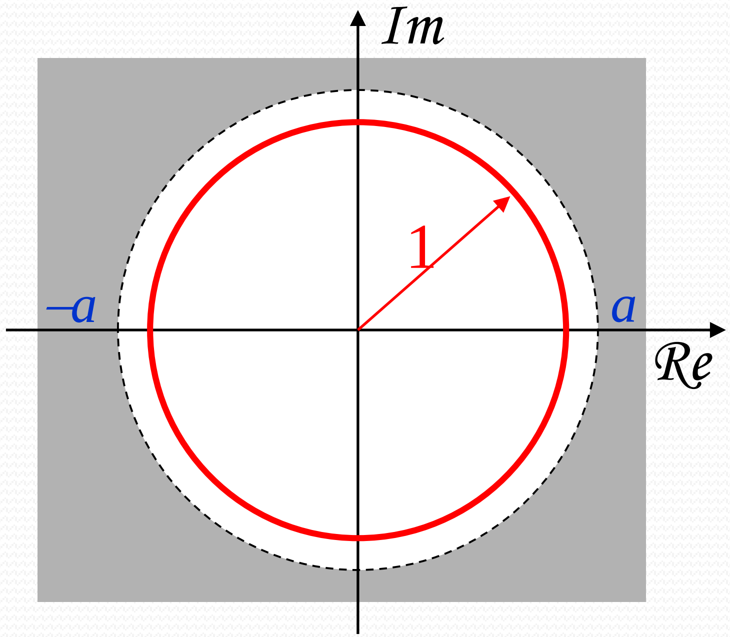
$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}$$

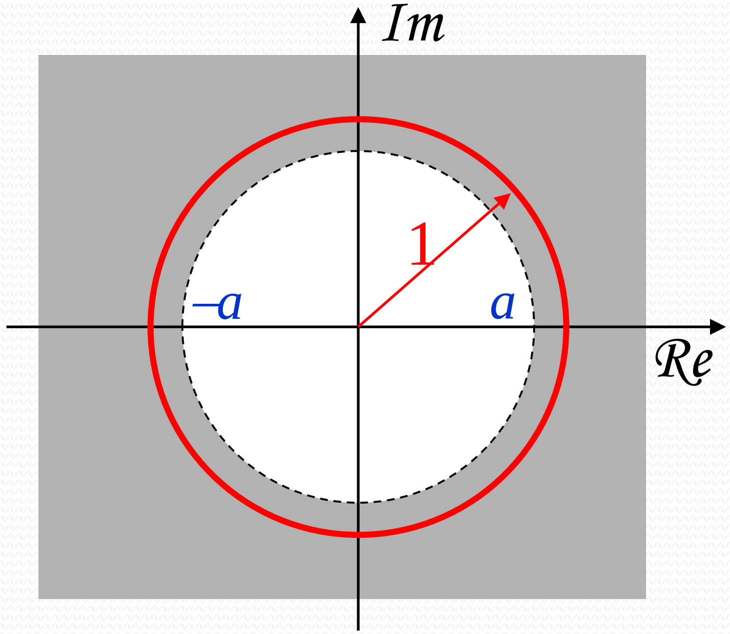
$$|z| > |a|$$

Example: A right sided Sequence ROC for $x(n)=a^n u(n)$

$$X(z)=\frac{z}{z-a}, \qquad |z|>|a|$$

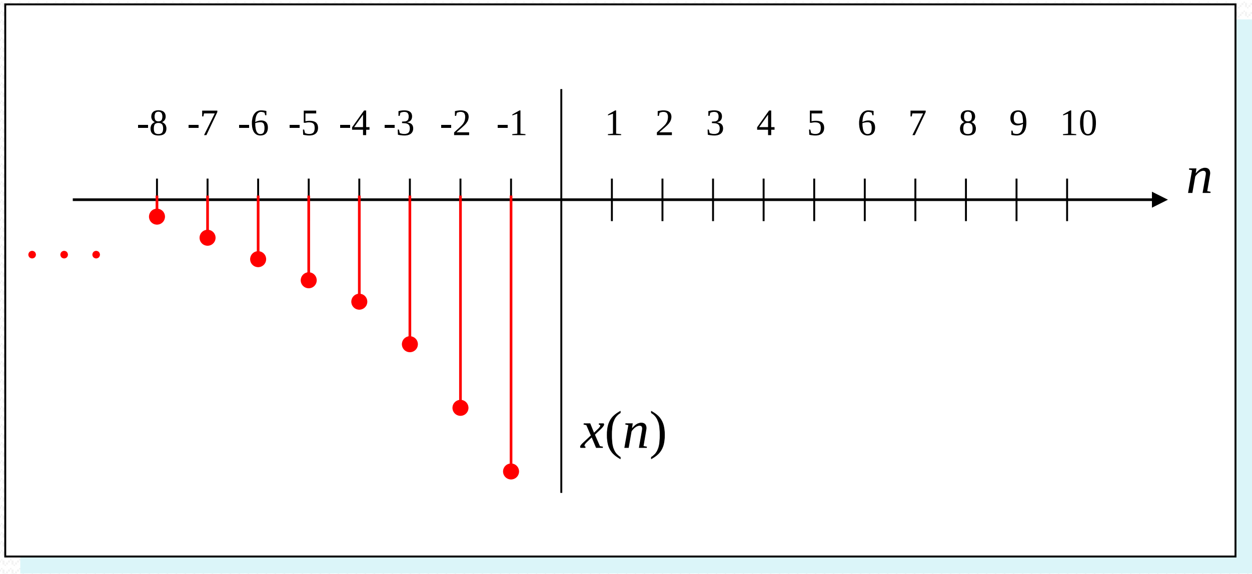
Which one is stable?





Example: A left sided Sequence

$$x(n) = -a^n u(-n-1)$$



Example: A left sided Sequence

$$x(n) = -a^n u(-n-1)$$

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n$$

$$= 1 - \sum_{n=0}^{\infty} a^{-n} z^n$$

For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} |a^{-1} z| < \infty \Rightarrow |a^{-1} z| < 1$$

$$\Rightarrow |z| < |a|$$

$$X(z) = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n = 1 - \frac{1}{1 - a^{-1} z} = \frac{z}{z - a}$$

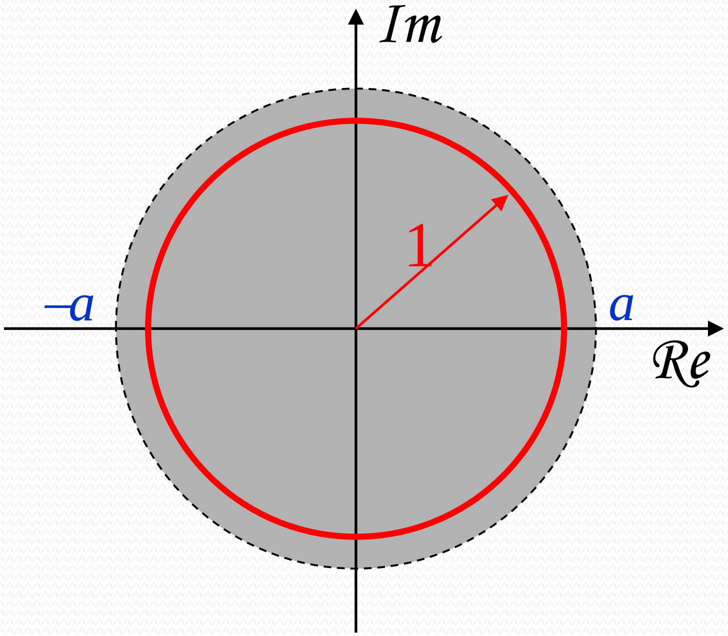
$$|z| < |a|$$

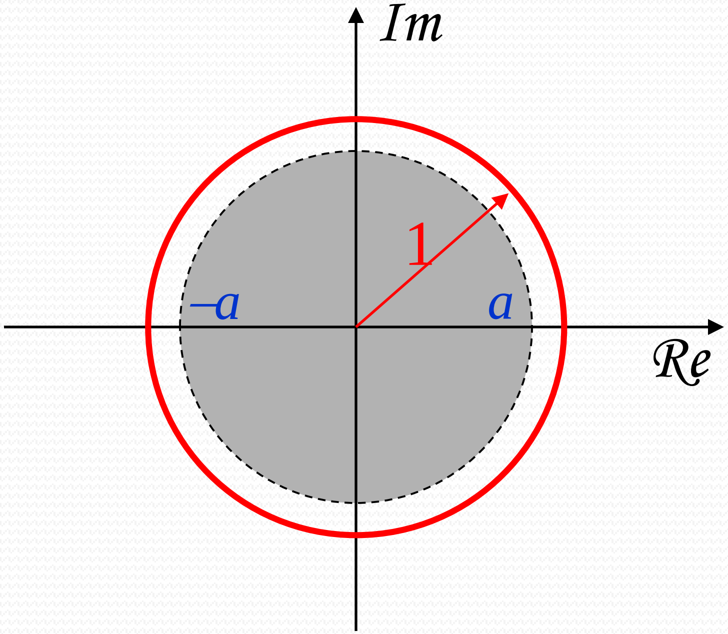
Example: A left sided Sequence

ROC for $x(n) = -a^n u(-n-1)$

$$X(z) = \frac{z}{z-a}, \qquad |z| < |a|$$

Which one is stable?





The z-Transform

Region of Convergence

Represent z-transform as a Rational Function

$$X(z) = \frac{P(z)}{Q(z)}$$

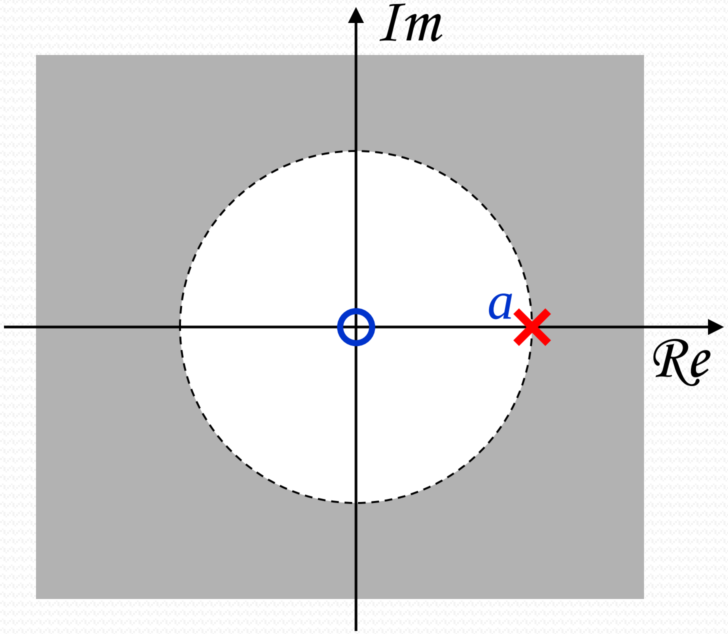
where $P(z)$ and $Q(z)$ are polynomials in z.

Zeros: The values of z's such that $X(z) = 0$

Poles: The values of z's such that $X(z) = \infty$

Example: A right sided Sequence

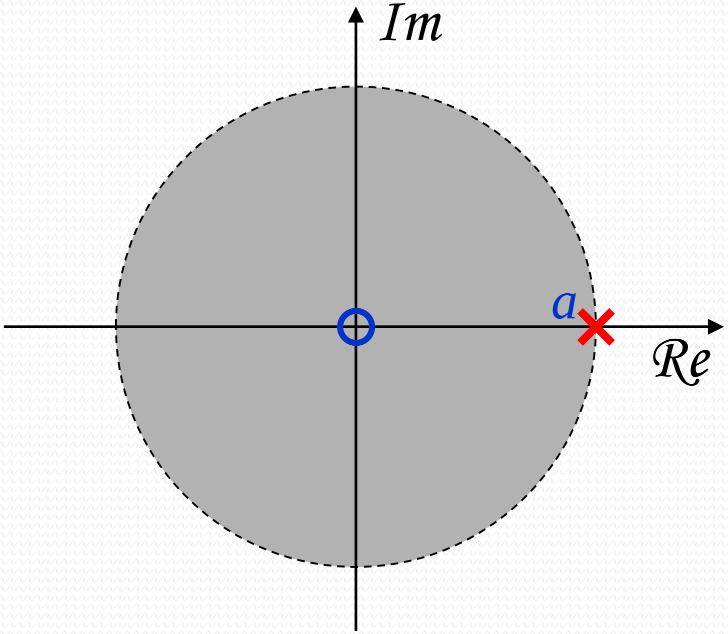
$$x(n) = a^n u(n) \Rightarrow X(z) = \frac{z}{z-a}, \qquad |z| > |a|$$



ROC is bounded by the pole and is the exterior of a circle.

Example: A left sided Sequence

$$x(n) = -a^n u(-n-1) \Rightarrow X(z) = \frac{z}{z-a}, \qquad |z| < |a|$$

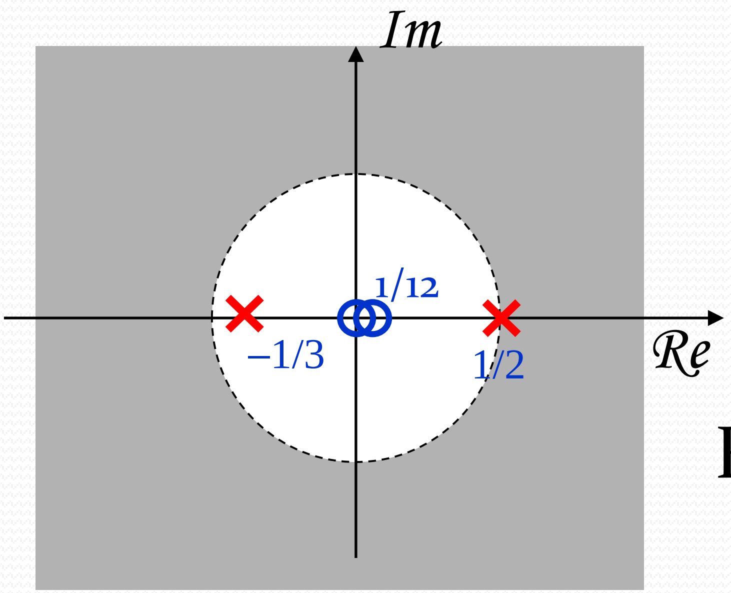


ROC is bounded by the pole and is the interior of a circle.

Example: Sum of Two Right Sided Sequences

$$x(n) = (\tfrac{1}{2})^n u(n) + (-\tfrac{1}{3})^n u(n)$$

$$\Rightarrow \quad X(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}} = \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}$$



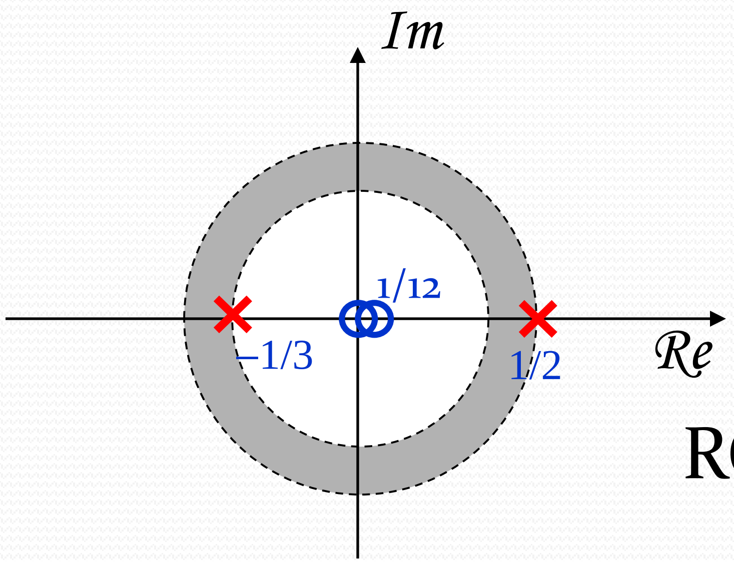
ROC is bounded by poles and is the exterior of a circle.

ROC does not include any pole

Example: A Two Sided Sequence

$$x(n) = (-\tfrac{1}{3})^n u(n) - (\tfrac{1}{2})^n u(-n-1)$$

$$\Rightarrow \quad X(z) = \frac{z}{z+\frac{1}{3}} + \frac{z}{z-\frac{1}{2}} = \frac{2z(z-\frac{1}{12})}{(z+\frac{1}{3})(z-\frac{1}{2})}$$



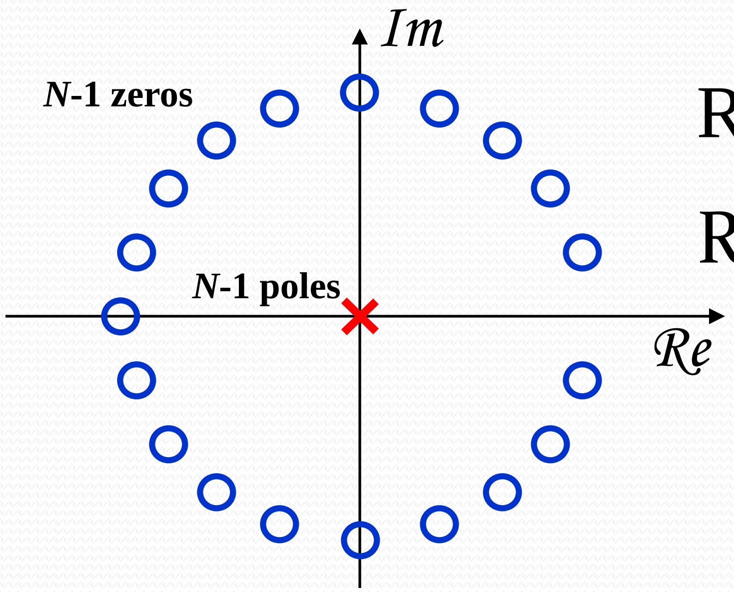
ROC is bounded by poles and is a ring.

ROC does not include any pole.

Example: A Finite Sequence

$$x(n) = a^n, \qquad 0 \le n \le N-1$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$



ROC: $0 < z < \infty$

ROC does not include any pole.

Always Stable

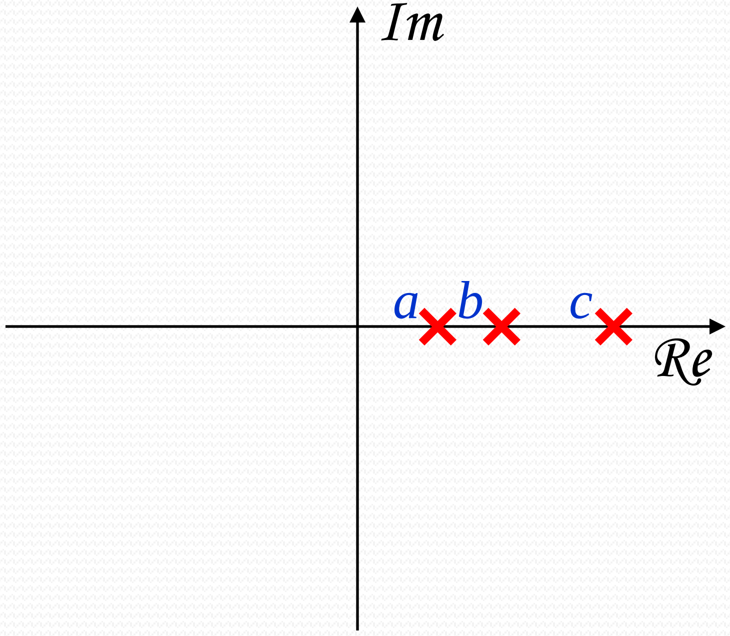
Properties of ROC

- A ring or disk in the z-plane centered at the origin.
- The Fourier Transform of $x(n)$ is converge absolutely iff the ROC includes the unit circle.
- The ROC cannot include any poles
- Finite Duration Sequences: The ROC is the entire z-plane except possibly z=0 or z=∞.
- Right sided sequences: The ROC extends outward from the outermost finite pole in $X(z)$ to z=∞.
- Left sided sequences: The ROC extends inward from the innermost nonzero pole in $X(z)$ to z=0.

More on Rational *z*-Transform

Consider the rational *z*-transform with the pole pattern:

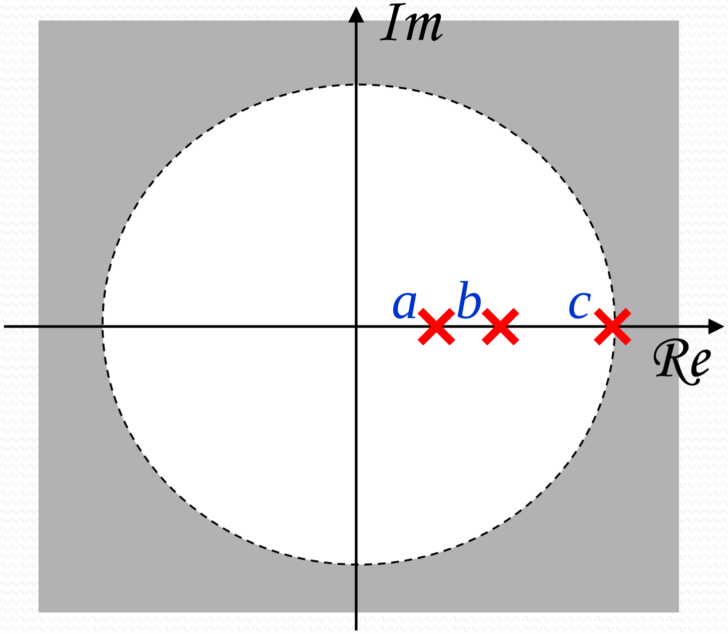
Find the possible ROC's



More on Rational *z*-Transform

Consider the rational *z*-transform with the pole pattern:

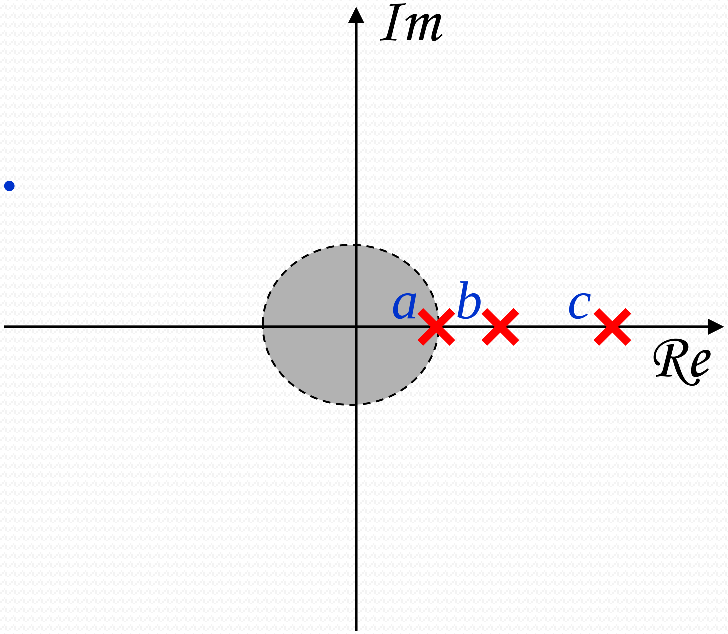
Case 1: A right sided Sequence.



More on Rational *z*-Transform

Consider the rational *z*-transform with the pole pattern:

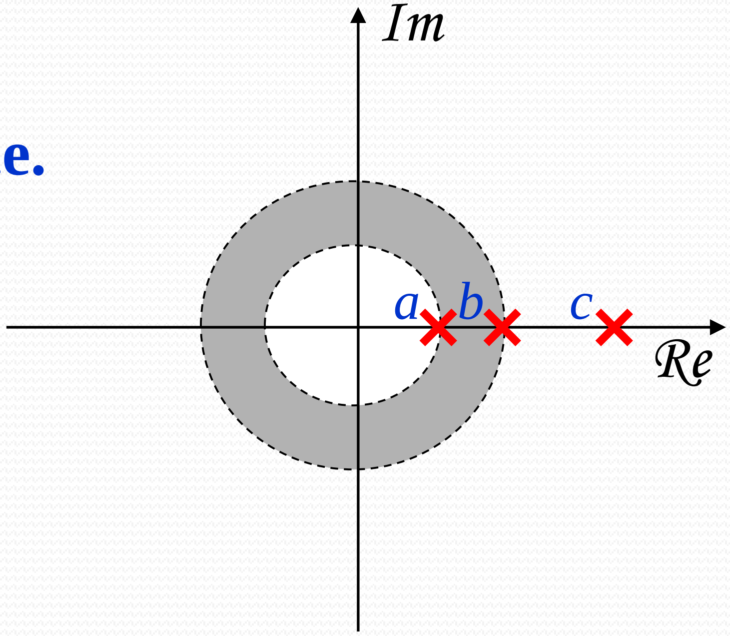
Case 2: A left sided Sequence.



More on Rational *z*-Transform

Consider the rational *z*-transform with the pole pattern:

Case 3: A two sided Sequence.



More on Rational *z*-Transform

Consider the rational *z*-transform with the pole pattern:

Case 4: Another two sided Sequence.